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Time Scales in Homogenization of Periodic Flows with Vanishing Molecular Diffusion

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We study the long time transport property of conservative systems perturbed by a small white noise. We introduce the dissipation and martingale times and show how they are related to the diffusion time on which a limit theorem is valid. The limit theorem is a probabilistic version of homogenization with vanishing molecular

1. INTRODUCTION

A continuous-time dynamical system on a certain state space is generally described by a differential equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{u}(\boldsymbol{x}(t)). \tag{1}$$

A dynamical system is called *conservative* if it leaves invariant certain regular measure on the state space. The typical examples considered here are dynamical systems on the Euclidean space R^d with the Lebesgue measure μ as an invariant measure. Thus, we assume in the present paper that $\boldsymbol{u}$ is a divergence-free vector field; i.e.,

$$\nabla \cdot \boldsymbol{u}(\mathbf{x}) = 0. \tag{2}$$

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We are interested in long time, large scale behavior of (1) on R^d that has certain repetitive basic structure and is under the influence of noise, namely,

$$dx^{\varepsilon}(t) = \boldsymbol{u}(x^{\varepsilon}(t))\,dt + \sqrt{\varepsilon}\,dw(t), \tag{3}$$

where $\boldsymbol{u}$ is either periodic, quasiperiodic, or random stationary, and $\boldsymbol{w}$ is the standard Brownian motion. The paramenter ε is small and is a measure of the magnitude of noise represented by $\sqrt{\varepsilon}\,w(t)$. The noise is chosen to have independent increments and preserve the Lebesgue measure μ. For simplicity, we shall be concerned only with the periodic case in the present paper; i.e., $\boldsymbol{u}(\mathbf{x})$ is a periodic function of $\mathbf{x} \in R^d$.

Physical systems modeled by Hamiltonian dynamics (a conservative system with the Lebesgue measure μ as an invariant measure on the phase space) are generally subject to noises arising from complex interactions. The local basic structure of the system may be represented as a reduced dynamics projected on the period cell, that is, $x(t)$ modulo the period. The projected dynamics (1) generally is not ergodic and, hence, has many invariant measures. Among them, the noise helps select a unique invariant measure for the noisy system (3). In the case that $\boldsymbol{u}$ is a random stationary process, the period cell is replaced with the ensemble space (abstract cell) and the Lebesgue measure μ with a translationally invariant probability measure.

In the case of fluid flows which will be the context of the present study, $\boldsymbol{u}(\mathbf{x})$ is the fluid velocity which has a repetitive basic structure such as closed streamlines, toroidal streamlines, and infinite streamlines. A diffusive particle in the fluid follows the trajectory determined by the fluid velocity $\boldsymbol{u}(\mathbf{x})$ and its own molecular diffusion $\sqrt{\varepsilon}\,w(t)$ which results from complex microscopic interactions with surrounding fluid molecules. The molecular diffusivity ε is usually very small.

What is the long time behavior of (3)? To answer this question, one needs to infer the global noisy Lagrangian dynamics from the local Eulerian information. The answer will depend on the time scale considered. The long time scale of interest is naturally expressed as a function λ_ε^2 of ε. We are then interested in the process $\lambda_\varepsilon^{-1} x^{\varepsilon}(\lambda_\varepsilon^2 t)$, as $\varepsilon \to 0$. Here we think of λ_ε^2 as time and t as a dimensionless multiplier. Different time scales are distinguished by different divergence rates of λ_ε^2 as ε tends to zero. A time scale $\lambda_{\varepsilon,1}^2$ is longer than another time scale $\lambda_{\varepsilon,2}^2$ if $\lim_{\varepsilon\to 0} \lambda_{\varepsilon,1}/\lambda_{\varepsilon,2} = \infty$. Two time scales are equivalent if neither of them is longer than the other.

When $\varepsilon = 0$, different kinds of trajectories generally coexist in the state space and there is no self-averaging unless the projected dynamics on the period cell is ergodic (see Dombre *et al.* [5] for a simple periodic flow exhibiting regular and irregular streamlines). Because ε is set to zero before a long time limit is taken, it may be considered as the *shortest* long time asymptotics of the noisy system, and it is a purely dynamical system problem with no simple answers.

On the other hand, if $\varepsilon > 0$ is held fixed while time is taken to infinity, the Lagrangian dynamics will eventually be dominated by noise and become purely diffusive with certain covariance matrix σ_ε dependent on ε. This is a result from homogenization theory (see [1]). The covariance matrix σ_ε is called the *effective diffusivity*. After homogenization, the small diffusion limit can be studied. Since time scale is taken to infinity before the small ε limit is considered, the corresponding results can be viewed as the longest time asymptotics of (3).

What is this small diffusion limit? This problem has been studied in detail for random as well as periodic flows (see [6–8, 13, 14]). In general, it is found that

$$\sigma_\varepsilon \asymp c^* \varepsilon^\alpha, \qquad \text{as} \quad \varepsilon \to 0$$

with the exponent α in the range

$$-1 \leqslant \alpha \leqslant 1$$

which depends on the velocity field $\boldsymbol{u}(\mathbf{x})$. Here and below $\asymp$ denotes the asymptotic equality as ε tends to zero. When $\alpha < 1$, the effective diffusivity is much larger than the molecular diffusivity ε and so is called *convection enhanced diffusion*.

The primary goal of the present paper is to study the asymptotics on the intermediate time scales between the shortest and the longest times, in particular, to determine the *diffusion time* scale on which the noisy process in (3) behaves effectively like the d-dimensional Brownian motion with σ_ε as the covariance. More specifically, we want to determine how quickly λ_ε^2 must diverge as $\varepsilon \to 0$ so that the nondimensional, scaled process

$$\lambda_\varepsilon^{-1} \sqrt{\sigma_\varepsilon^{-1}}\, \boldsymbol{x}^\varepsilon(\lambda_\varepsilon^2 t)$$

converges to standard Brownian motion as $\varepsilon \to 0$. Let $\mathscr{T}_\varepsilon$ be the set of t_ε such that the convergence to the Brownian motion holds for all $\lambda_\varepsilon \geqslant t_\varepsilon$. The *diffusion time* t_{diff} may then be defined as the infimum of $\mathscr{T}_\varepsilon$.

We approach this problem by decomposing the perturbed process into a martingale $\rho^\varepsilon(\boldsymbol{x}^\varepsilon(t))$ and a nearly periodic fluctuation $\chi^\varepsilon(\boldsymbol{x}^\varepsilon(t))$

$$\boldsymbol{x}^\varepsilon(t) = \rho^\varepsilon(\boldsymbol{x}^\varepsilon(t)) - \chi^\varepsilon(\boldsymbol{x}^\varepsilon(t)), \tag{4}$$

where $\chi^\varepsilon = \{\chi_i^\varepsilon(\mathbf{x})\}$ is a zero mean vector-valued function periodic in $\mathbf{x}$ and $\rho^\varepsilon(\mathbf{x}) = \{\rho_i^\varepsilon(\mathbf{x})\}$ is harmonic with respect to the generator $\mathscr{L}^\varepsilon$

$$\mathscr{L}^\varepsilon \rho_i^\varepsilon = \varepsilon \, \Delta \rho_i^\varepsilon + \boldsymbol{u} \cdot \nabla \rho_i^\varepsilon = 0, \qquad \forall i = 1, ..., d$$

subject to certain boundary conditions. This technique has been used by Papanicolaou *et al.* [11] and Osada [10] in another context.

The martingale $\rho^\varepsilon(\boldsymbol{x}^\varepsilon(t))$ dominates the effective behavior over long times and becomes a Brownian motion with the effective diffusivity matrix σ_ε given by

$$(\sigma_\varepsilon)_{ij} = \varepsilon \langle \nabla \rho_i^\varepsilon \cdot \nabla \rho_j^\varepsilon \rangle, \qquad \forall i, j. \tag{5}$$

We note that this definition of effective diffusivity accounts only for the symmetric part of the full effective tensor defined in, for example, [8]. Here and below $\langle \cdot \rangle$ denotes the volume average with respect to the Lebesgue measure μ.

For this long time asymptotics to be valid, the time scale first needs to be longer than what we call the *dissipation* and the *martingale* time scales so that the fluctuation $\chi(\boldsymbol{x}^\varepsilon(t))$ dies out. The martingale time t_{mart} is the ratio of the variance of fluctuation and the effective diffusivity:

$$t_{mart} = \sup_i \frac{\langle (\chi_i^\varepsilon)^2 \rangle}{\sigma_\varepsilon(\boldsymbol{e}_i)}. \tag{6}$$

The dissipation time t_{diss} is defined by

$$\|\mathscr{P}_{t_{diss}}^\varepsilon\|_{2 \to 2} = \tfrac{1}{2}, \tag{7}$$

where $\mathscr{P}_t^\varepsilon$ is the semigroup generated by $\mathscr{L}^\varepsilon$ and $\|\cdot\|_{2\to 2}$ is the operator from L_0^2 to L_0^2, the space of square-integrable, mean zero, periodic functions. There is one and only one such t_{diss} satisfying (7) because the operator norm $\|\mathscr{P}_t\|_{2\to 2}$ is strictly decreasing for any $\varepsilon > 0$. We define the decay rate function $N_\varepsilon(t)$ of the semigroup $\mathscr{P}_t^\varepsilon$ by

$$\|\mathscr{P}_t^\varepsilon\|_{2\to 2} = e^{-N_\varepsilon(t)}. \tag{8}$$

By (7) and (8), we have $N_\varepsilon(t_{diss}) = \ln 2$.

We shall prove

THEOREM 1. *Let $\boldsymbol{u}(\mathbf{x})$ be a continuously differentiable, divergence free periodic vector field with zero mean, $\langle \boldsymbol{u} \rangle = 0$, in dimension d. Suppose that the time scale λ_ε^2 satisfies*

$$\lambda_\varepsilon^2 \gg t_{mart} \vee t_{diss} \vee \sup_i \sigma_\varepsilon^{-1}(\boldsymbol{e}_i) \tag{9}$$

and

$$\lim_{\varepsilon\to 0}\left\{N_\varepsilon(\lambda_\varepsilon^2)+\left(\frac{d}{2}+1\right)\log(\lambda_\varepsilon^2\varepsilon)\right\}=+\infty. \tag{10}$$

Let the initial point $\boldsymbol{x}^\varepsilon(0)$ *be distributed uniformly in a fixed period cell. Then the family of processes*

$$\mathsf{X}^\varepsilon(t):=\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}\,\boldsymbol{x}^\varepsilon(\lambda_\varepsilon^2 t),\qquad t\in[T_0,T],\quad \forall 0<T_0<T<\infty \tag{11}$$

converges weakly, as $\varepsilon\to 0$, *to the d-dimensional standard Brownian motion* $\mathbf{B}(t)$, $t\in[T_0,T]$, $\mathbf{B}(0)=0$.

The requirement $\lambda_\varepsilon^2\gg\sup_i\,\sigma_\varepsilon^{-1}(e_i)$ in (9) ensures that the spatial scale, as well as the time scale, is large.

By the probabilistic representation of the semigroup

$$\mathscr{P}_t^\varepsilon f(\mathbf{x})=\mathbf{E}_{\mathbf{x}}\{f(\boldsymbol{x}^\varepsilon(t))\},\qquad \boldsymbol{x}^\varepsilon(0)=\mathbf{x}$$

we can restate the result of the theorem in terms of the solution of the evolution equation

$$\frac{\partial\phi(t,\mathbf{x})}{\partial t}=\mathscr{L}^\varepsilon\phi(t,\mathbf{x}),\qquad \phi(0,\mathbf{x})=f(\mathbf{x})\in C_0^\infty \tag{12}$$

as follows. Rescaling Eq. (12), we have the evolution equation for the rescaled process (11)

$$\frac{\partial\phi^\varepsilon(t,\mathbf{x})}{\partial t}=\frac{\varepsilon}{2}\nabla\cdot\sigma_\varepsilon^{-1}\,\nabla\phi^\varepsilon(t,\mathbf{x})+\lambda_\varepsilon\sqrt{\sigma_\varepsilon^{-1}}\,\boldsymbol{u}(\lambda_\varepsilon\sqrt{\sigma_\varepsilon}\,\mathbf{x})$$
$$\cdot\nabla\phi^\varepsilon(t,\mathbf{x}),\qquad \phi^\varepsilon(0,\mathbf{x})=f(\mathbf{x})\in C_0^\infty(R^d).$$

The theorem implies that, as ε tends to zero, the $\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}$ cell-averaged solution, $\langle\phi^\varepsilon(t,\mathbf{x})\rangle_\varepsilon$, converges pointwise to the solution $\bar\phi(t,\mathbf{x})$ of the equation

$$\frac{\partial\bar\phi(t,\mathbf{x})}{\partial t}=\frac{1}{2}\Delta\bar\phi(t,\mathbf{x}),\qquad \bar\phi(0,\mathbf{x})=f(\mathbf{x}).$$

As a result of Theorem 1.1 one has the upper bound on the (average) diffusion time

$$t_{diff}\lesssim\frac{1}{\varepsilon}\wedge\left(t_{mart}\vee t_{diss}\log\frac{1}{\varepsilon}\right)$$

(see Proposition 5).

Theorem 1 is different from the usual central limit theorem in that the scaling is nondiffusive due to possible blow-up or vanishing of the effective diffusivity σ_ε. In particular, one can interpret the scaling in (11) as *super-diffusive* when $\sigma_\varepsilon \to \infty$ and as *subdiffusive* when $\sigma_\varepsilon \to 0$ as ε tends to zero (see examples in Section 6). Since λ_ε can be any function of ε satisfying (9) and (10) while σ_ε is a *fixed* function of ε, the anomalous scaling depends on the time scale of consideration and *the longer the time scale the less anomalous the scaling is*. This points to the subtlety in interpreting experimental or numerical data for scaling behaviors in noisy systems.

To make Theorem 1 more concrete, one has to determine the asymptotics of t_{diss}, t_{mart} and σ_ε as ε tends to zero as we will demonstrate in Sections 5 and 6. All of the above quantities are determined by the velocity field $\boldsymbol{u}(\mathbf{x})$.

The difficulty in extending the result of Theorem 1 to the processes starting at a *fixed* initial point may be seen in the following estimate used in the proof of Theorem 1.

LEMMA 1. *Let $\boldsymbol{u}(\mathbf{x})$ be a continuously differentiable, divergence free vector field on the torus. Then the semigroup $\mathscr{P}_t^\varepsilon$ and its adjoint $\mathscr{P}_t^{\varepsilon *}$ satisfy*

$$\|\mathscr{P}_t^\varepsilon\|_{1\to\infty} \leqslant \frac{C_\delta}{(\varepsilon t)^{d/2}}\, e^{-N_\varepsilon(\delta t)}, \qquad \forall 0<\delta<1, \qquad \forall t>0 \tag{13}$$

$$\|\mathscr{P}_t^{\varepsilon *}\|_{1\to\infty} \leqslant \frac{C_\delta}{(\varepsilon t)^{d/2}}\, e^{-N_\varepsilon(\delta t)}, \qquad \forall 0<\delta<1, \qquad \forall t>0 \tag{14}$$

with the rate function N_ε as given by (8) *and some positive constant C_δ. Here $\|\cdot\|_{1\to\infty}$ denotes the operator norm from L_0^1, the space of integrable, mean-zero, periodic functions, to L_0^∞, the space of bounded, mean-zero, periodic functions.*

The presence of, possibly long, transient behaviors is indicated by the short-time singularity $(\varepsilon t)^{-d/2}$ in (13) which also gives rise to the condition (10). The proof of Lemma 1 is given in Section 3.

When $\langle \boldsymbol{u}\rangle \neq 0$, the mean drift has to be subtracted first before one considers the corresponding limit theorem. This amounts to changing to the moving frame of the mean drift and turns the velocity time dependent. Transport in time-periodic velocity fields requires a different treatment as the semi-group technique is not available. One approach is to consider the time-1 map of such a system followed by a convolution with a heat kernel. A limit theorem analogous to Theorem 1 is fully expected to hold. The techniques are rather different and will be published elsewhere. We will briefly discuss one such example in Section 6.

It is worth noting that there are numerical evidences of strong anomalous diffusion in the *absence* of molecular diffusion in the sense of non-Gaussian limit (see, e.g., [2]). Theorem 1 says that the Gaussian limit is restored with arbitrarily small molecular diffusion but sufficiently long time. The anomaly does not, however, completely disappear, but manifests itself as singular behaviors in the gradient of the corrector as $\varepsilon \to 0$.

In the following c, c', ... denote constants independent of ε.

2. CORRECTOR, MARTINGALE AND MARTINGALE TIMES

For simplicity, we use as the basis for R^d the set $\{e_i\}_{i=1}^d$ of orthonormal eigenvectors of the effective diffusivity matrix σ_ε. In general, $\{e_i\}$ may be a function of ε.

First we assume that $\boldsymbol{u}$ has zero mean $\langle \boldsymbol{u} \rangle = 0$. We decompose the sample path $\boldsymbol{x}_\varepsilon(t)$ into a martingale $\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ and a fluctuation $\chi^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ as in (4). More precisely, let $\chi^\varepsilon(\mathbf{x}) = (\chi_1^\varepsilon(\mathbf{x}), ..., \chi_d^\varepsilon(\mathbf{x}))$ be the zero mean, periodic solution of the equation

$$\left(\frac{\varepsilon}{2}\Delta + \boldsymbol{u} \cdot \nabla\right)\chi^\varepsilon = -\boldsymbol{u}. \tag{15}$$

The solvability of (15) is guaranteed by $\langle \boldsymbol{u} \rangle = 0$. Let

$$\rho^\varepsilon(\mathbf{x}) = \mathbf{x} + \chi^\varepsilon(\mathbf{x}). \tag{16}$$

Then we have, after simple manipulations, that

$$\mathscr{L}^\varepsilon \rho^\varepsilon = 0 \tag{17}$$

and ρ^ε itself is not periodic but has a periodic gradient with the mean

$$\langle \nabla \rho^\varepsilon \rangle = I.$$

The function χ^ε is called the corrector in homogenization theory. Its significance is manifest in the identity (see [1, 7])

$$\sigma_\varepsilon(\boldsymbol{e}_i) \equiv (\sigma_\varepsilon)_{ii} = \varepsilon + \varepsilon \langle |\nabla \chi_i^\varepsilon|^2 \rangle = \varepsilon \langle |\nabla \rho_i^\varepsilon|^2 \rangle, \qquad \forall i, \tag{18}$$

where $\sigma_\varepsilon(\boldsymbol{e}_i)$ is the *effective diffusivity* in the direction $\boldsymbol{e}_i$. We note that (18) is only the symmetric part of the full effective tensor considered in [7].

From (17) it follows that $\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ is a martingale with the quadratic variation, $|\nabla \rho^\varepsilon|^2\,(\mathsf{X}_\varepsilon(t))$. Hence (18) indicates that if the fluctuation $\chi^\varepsilon(\boldsymbol{x}_\varepsilon(t))$

dies out for a sufficiently long time, then the effective diffusivity will be given by the mean quadratic variation of the martingale $\rho^{\varepsilon}(\boldsymbol{x}_{\varepsilon}(t))$. Normalizing the process by rescaling (4), we have

$$\lambda_{\varepsilon}^{-1}\sqrt{\sigma_{\varepsilon}^{-1}}\cdot \boldsymbol{x}^{\varepsilon}(\lambda_{\varepsilon}^{2}t)=\lambda_{\varepsilon}^{-1}\sqrt{\sigma_{\varepsilon}^{-1}}\,\rho^{\varepsilon}(\boldsymbol{x}^{\varepsilon}(\lambda_{\varepsilon}^{2}t))-\lambda_{\varepsilon}^{-1}\sqrt{\sigma_{\varepsilon}^{-1}}\,\chi^{\varepsilon}(\boldsymbol{x}^{\varepsilon}(\lambda_{\varepsilon}^{2}t)). \tag{19}$$

The preceding analysis suggests the definition (6) of the *martingale time* t_{mart}. On the time scale longer than the martingale time the second moment of the fluctuation in (19) diminishes in the limit.

The following result for σ_{ε} follows from the energy estimate for (15):

PROPOSITION 1. *For any continuously differentiable, divergence free vector field* $\mathbf{u}(\mathbf{x})$ *on the torus, there exists a constant c such that*

$$\varepsilon \leqslant \sigma_{\varepsilon}(e_i) \leqslant c/\varepsilon, \qquad \forall i$$

for sufficiently small $\varepsilon > 0$.

Proof. Multiplying (15) by χ^{ε} and integrating by parts we obtain

$$\varepsilon\langle \nabla\chi_i^{\varepsilon}\cdot\nabla\chi_i^{\varepsilon}\rangle = \langle u_i\chi_i^{\varepsilon}\rangle, \qquad \forall i.$$

The term $\langle \chi_i^{\varepsilon}\boldsymbol{u}\cdot\nabla\chi_i^{\varepsilon}\rangle$ drops out because of the divergence free condition (2). Using the Cauchy–Schwarz and the Poincare inequalities, we have that

$$\varepsilon\langle \nabla\chi_i^{\varepsilon}\cdot\nabla\chi_i^{\varepsilon}\rangle \leqslant \sqrt{\langle u_i^2\rangle}\sqrt{\langle (\chi_i^{\varepsilon})^2\rangle} \leqslant \sqrt{\langle u_i^2\rangle}\sqrt{c\langle (\nabla\chi_i^{\varepsilon})^2\rangle}$$

and hence

$$\varepsilon\langle \nabla\chi_i^{\varepsilon}\cdot\nabla\chi_i^{\varepsilon}\rangle \leqslant \frac{c}{\varepsilon}\langle u_i^2\rangle.$$

This proves the upper bound. The lower bound is obvious in view of (18). ∎

When the mean drift $\langle \boldsymbol{u}(\mathbf{x})\rangle \neq 0$, the dieanalysis needs to be modified as follows (see [7] for details). Write $\mathbf{u}(\mathbf{x}) = \langle \boldsymbol{u}\rangle + \boldsymbol{u}'(\mathbf{x})$ where $\boldsymbol{u}'(\mathbf{x})$ is the zero-mean field. Let $\chi^{\varepsilon}(\mathbf{x})$ be the periodic solution of

$$\frac{\varepsilon}{2}\Delta\chi^{\varepsilon}+\boldsymbol{u}\cdot\nabla\chi^{\varepsilon}+\boldsymbol{u}'=0$$

which is solvable by the Fredholm alternative condition $\langle \boldsymbol{u}' \rangle = 0$. Let

$$\rho^{\varepsilon}(\mathbf{x}) = \mathbf{x} + \chi^{\varepsilon}(\mathbf{x}).$$

The effective diffusivity of the relative displacement $\frac{1}{\lambda_\varepsilon}(\boldsymbol{x}^{\varepsilon}(\lambda_\varepsilon^2 t) - \lambda_\varepsilon^2 \langle \boldsymbol{u} \rangle t)$ as $\lambda_\varepsilon \to \infty$ is given again by (18).

3. THE ULTRACONTRACTIVITY ESTIMATE: PROOF OF LEMMA 1

An important fact for this section is that the adjoint of $\mathscr{L}^{\varepsilon}$, due to the divergence free condition $\nabla \cdot \boldsymbol{u} = 0$, is

$$\mathscr{L}^{\varepsilon *} = \frac{\varepsilon}{2} \Delta - \boldsymbol{u}(\mathbf{x}) \cdot \nabla$$

which is the formal generator of the adjoint semigroup $\mathscr{P}_t^{\varepsilon *}$ of $\mathscr{P}_t^{\varepsilon}$.

First, we show

LEMMA 2. *The semigroup $\mathscr{P}_t^{\varepsilon}$ and its adjoint $\mathscr{P}_t^{\varepsilon *}$ satisfy*

$$\|\mathscr{P}_t^{\varepsilon}\|_{1 \to 2} \leqslant \frac{1}{c} (\varepsilon t)^{-\frac{d}{4}}, \qquad \forall t > 0 \tag{20}$$

$$\|\mathscr{P}_t^{\varepsilon *}\|_{1 \to 2} \leqslant \frac{1}{c} (\varepsilon t)^{-\frac{d}{4}}, \qquad \forall t > 0, \tag{21}$$

where $\|\cdot\|_{1 \to 2}$ is the operator norm from L_0^1 to L_0^2.

Proof of Lemma 2. To show (20)–(21), we adapt Nash's method (see [15]) to the context of asymmetric process.

Let φ be an integrable, mean-zero, periodic function. Consider the identity

$$\frac{d}{dt} \langle \mathscr{P}_t^{\varepsilon} \varphi \cdot \mathscr{P}_t^{\varepsilon} \varphi \rangle = 2 \langle \mathscr{L}^{\varepsilon} \mathscr{P}_t^{\varepsilon} \varphi \cdot \mathscr{P}_t^{\varepsilon} \varphi \rangle = \varepsilon \langle \Delta \mathscr{P}_t^{\varepsilon} \varphi \cdot \mathscr{P}_t^{\varepsilon} \varphi \rangle \tag{22}$$

$$= -\varepsilon \langle \nabla \mathscr{P}_t^{\varepsilon} \varphi \cdot \nabla \mathscr{P}_t^{\varepsilon} \varphi \rangle$$

which, in terms of the notation $\varphi_t \equiv \mathscr{P}_t^{\varepsilon} \varphi$, is

$$\frac{d}{dt} \langle \varphi_t \cdot \varphi_t \rangle = -\varepsilon \langle \nabla \varphi_t \cdot \nabla \varphi_t \rangle. \tag{23}$$

In view of the Nash inequality

$$\|\varphi_t\|_2^{2+\frac{4}{d}} \leqslant c\, \|\nabla \varphi_t\|_2^2\, \|\varphi_t\|_1^{\frac{4}{d}},$$

we obtain from Eq. (23)

$$\frac{dt}{\|\varphi_t\|_1^{\frac{4}{d}}} \leqslant -\frac{c}{\varepsilon}\, \frac{d\, \|\varphi_t\|_2^2}{\|\varphi_t\|_2^{2+\frac{4}{d}}} \tag{24}$$

After integration, (24) becomes

$$\|\varphi_t\|_2^2 \leqslant \left(\frac{\varepsilon}{c} \int_0^t \|\varphi_s\|_1^{-\frac{4}{d}}\, ds + \|\varphi\|_2^{-\frac{4}{d}}\right)^{-d/2}, \qquad \forall t \geqslant 0. \tag{25}$$

Now that $\mathscr{P}_t^\varepsilon$ is contractive on L^p for $1 \leqslant p \leqslant \infty$, in particular,

$$\|\varphi_t\|_1 \leqslant \|\varphi\|_1, \qquad \forall t \geqslant 0$$

we immediately have from (25)

$$\|\varphi_t\|_2^2 \leqslant \frac{c\, \|\varphi\|_1^2}{(\varepsilon t)^{\frac{d}{2}}}, \qquad \forall t \geqslant 0 \tag{26}$$

or

$$\|\mathscr{P}_t^\varepsilon\|_{1\to 2} \leqslant \frac{c}{(\varepsilon t)^{d/4}}.$$

Note that the preceding estimates rely only on the skew adjointness of the convection operator $\boldsymbol{u} \cdot \nabla$. So the estimates (25) and (26) also hold for $\varphi_t'^* \equiv \mathscr{P}_t^{\varepsilon *} \varphi$.

Let us complete the proof of Lemma 1.

By the semigroup property we have

$$\begin{aligned} \|\mathscr{P}_t^\varepsilon\|_{1\to\infty} &\leqslant \|\mathscr{P}^\varepsilon_{\frac{(1-\delta)}{2}t}\|_{1\to 2}\, \|\mathscr{P}^\varepsilon_{\delta t}\|_{2\to 2}\, \|\mathscr{P}^\varepsilon_{\frac{(1-\delta)}{2}t}\|_{2\to\infty} \\ &= \|\mathscr{P}^\varepsilon_{\frac{(1-\delta)}{2}t}\|_{1\to 2}\, \|\mathscr{P}^\varepsilon_{\delta t}\|_{2\to 2}\, \|\mathscr{P}^{\varepsilon *}_{\frac{(1-\delta)}{2}t}\|_{1\to 2}. \end{aligned} \tag{27}$$

By Lemma 2, we get from (27) that

$$\|\mathscr{P}_t^\varepsilon\|_{1\to\infty} \leqslant \frac{c}{((1-\delta)\, \varepsilon t)^{d/2}}\, e^{-N_\varepsilon(\delta t)},$$

where $N_\varepsilon(t)$ is the decay rate function for $\|\mathscr{P}_t^\varepsilon\|_{2\to 2}$ as defined in (8). Hence the constant C_δ in the theorem diverges like $(1-\delta)^{-d/2}$ as δ tends to one.

By duality, $\|\mathscr{P}_t^\varepsilon\|_{1\to\infty} = \|\mathscr{P}_t^{\varepsilon *}\|_{1\to\infty}$; the adjoint semigroup $\mathscr{P}_t^{\varepsilon *}$ satisfies the same bound.

4. PROOF OF THEOREM 1

First we note that the fluctuation $\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}\,\chi^\varepsilon(x^\varepsilon(t))$ is negligible because

$$\lim_{\varepsilon\to 0}\frac{1}{\lambda_\varepsilon^2}\sigma_\varepsilon^{-1}(e_i)\,\mathbf{E}[|\chi_i^\varepsilon|^2\,(x^\varepsilon(\lambda_\varepsilon^2 t))] = \lim_{\varepsilon\to 0}\frac{1}{\lambda_\varepsilon^2}\sigma_\varepsilon^{-1}(e_i)\langle|\chi_i^\varepsilon|^2\rangle = 0, \qquad \forall i. \tag{28}$$

So, for the proof, the process $\mathsf{X}^\varepsilon(t)$ and the martingale

$$\mathsf{Y}^\varepsilon(t) \equiv \lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}\,\rho^\varepsilon(x^\varepsilon(\lambda_\varepsilon^2 t))$$

can be used interchangeably. Let $\mathsf{Y}_i^\varepsilon(t)$ be the ith component of $\mathsf{Y}^\varepsilon(t)$ and we have the formula

$$\mathsf{Y}_i^\varepsilon(t) = \mathsf{Y}_i^\varepsilon(0) + \int_0^t \sqrt{\varepsilon\sigma_\varepsilon^{-1}}\,\nabla\rho_i^\varepsilon(x^\varepsilon(\lambda_\varepsilon^2\tau))\,dw(\tau). \tag{29}$$

Note that the initial points $\mathsf{X}^\varepsilon(0)$ are convergent:

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}\,x^\varepsilon(0) = 0, \qquad \text{a.s.}$$

which follows from

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon^2\sigma_\varepsilon(e_i) = \infty, \qquad \forall i. \tag{30}$$

Now we check against a criterion for tightness [9, p. 64] First, the random variable $\mathsf{X}^\varepsilon(t)$ for any fixed $t>0$ is tight. This follows from

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbf{E}\{|\mathsf{Y}^\varepsilon(t)|^2\} &= \lim_{\varepsilon\to 0}\langle|\mathsf{Y}^\varepsilon(0)|^2\rangle + \lim_{\varepsilon\to 0}\int_0^t \varepsilon\langle|\sqrt{\sigma_\varepsilon^{-1}}\,\nabla\rho^\varepsilon|^2\rangle\,d\tau\\ &= dt < \infty,\end{aligned} \tag{31}$$

where $\mathbf{E}$ is the expectation w.r.t. the product of the Wiener measure and the Lebesgue measure. Here we have used (29). Note that the above calculation also shows that the limiting process, if it exists, must have the total variance dt at time $t>0$.

Second, there exists a constant $C>0$ such that

$$\limsup_{\varepsilon\to 0}\mathbf{E}\{|\mathsf{X}^\varepsilon(t)-\mathsf{X}^\varepsilon(s)|^4\} \leqslant C\,|t-s|^2, \quad \forall t, s\in[T_0, T],\ 0<T_0<T. \tag{32}$$

By (29) we have

$$\mathbf{E}_{\mathbf{x}}\{|\mathsf{Y}_i^\varepsilon(t)-\mathsf{Y}_i^\varepsilon(s)|^4\}=3\left(\int_s^t \varepsilon\sigma_\varepsilon^{-1}\mathbf{E}_{\mathbf{x}}\{|\nabla\rho_i^\varepsilon(\mathbf{x}^\varepsilon(\lambda_\varepsilon^2\tau))|^2\}\,d\tau\right)^2.$$

Thus,

$$\begin{aligned}\limsup_{\varepsilon\to0}\ \mathbf{E}\{|\mathsf{X}_i^\varepsilon(t)-\mathsf{X}_i^\varepsilon(s)|^4\}&=\limsup_{\varepsilon\to0}\ \mathbf{E}\{|\mathsf{Y}_i^\varepsilon(t)-\mathsf{Y}_i^\varepsilon(s)|^4\}\\&=3\limsup_{\varepsilon\to0}\ \left\langle\left(\int_s^t \varepsilon\sigma_\varepsilon^{-1}\mathbf{E}_{\mathbf{x}}\{|\nabla\rho_i^\varepsilon(\mathbf{x}^\varepsilon(\lambda_\varepsilon^2\tau))|^2\}\,d\tau\right)^2\right\rangle\\&=3(t-s)^2,\qquad \forall t,s\in[T_0,T],\quad 0<T_0<T\end{aligned}$$

by Lemma 1 and the dominated convergence theorem. And the tightness of $\mathsf{X}^\varepsilon(t)$ follows.

To identify the limit we consider the martingale

$$M_t^\varepsilon=f(\mathsf{Y}_i^\varepsilon(t))-\int_0^t\frac{1}{2}\,\varepsilon\sum_i\sigma_\varepsilon^{-1}(e_i)\,|\nabla\rho_i^\varepsilon(\mathbf{x}^\varepsilon(\lambda_\varepsilon^2\tau))|^2\,\frac{\partial^2}{\partial x_i^2}f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau$$

for any $f\in C^\infty(R^d)$. We have

$$\begin{aligned}&f(\mathsf{Y}_i^\varepsilon(t))-\int_0^t\frac{1}{2}\Delta f(\mathsf{Y}_i^\varepsilon(s))\,ds\\&\quad=\frac{1}{2}\int_0^t\left[\varepsilon\sum_i\sigma_\varepsilon^{-1}(e_i)\,|\nabla\rho_i^\varepsilon(\mathbf{x}^\varepsilon(\lambda_\varepsilon^2\tau))|^2\,\frac{\partial^2}{\partial x_i^2}-\Delta\right]f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau+M_t^\varepsilon.\end{aligned}$$

Taking the conditional expectation w.r.t. the filtration $\mathscr{F}_s$ of events up to time s, $0<s<t$, we obtain

$$\begin{aligned}&\mathbf{E}\left\{f(\mathsf{Y}_i^\varepsilon(t))-\frac{1}{2}\int_0^t\Delta f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau\,\Big|\,\mathscr{F}_s\right\}\\&\quad=f(\mathsf{Y}_i^\varepsilon(s))-\frac{1}{2}\int_0^s\Delta f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau\\&\qquad+\frac{1}{2}\int_s^t\mathbf{E}\left\{\sum_i\left[\varepsilon\sigma_\varepsilon^{-1}(e_i)\,|\nabla\rho_i^\varepsilon(\mathbf{x}^\varepsilon(\lambda_\varepsilon^2\tau))|^2-1\right]\frac{\partial^2}{\partial x_i^2}f(\mathsf{Y}_i^\varepsilon(\tau))\,\Big|\,\mathscr{F}_s\right\}d\tau.\end{aligned}$$

We claim that

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_s^t \mathbf{E}\left\{\sum_i \left[\varepsilon\sigma_\varepsilon^{-1}(e_i)\,|\nabla\rho_i^\varepsilon(x^\varepsilon(\lambda_\varepsilon^2\tau))|^2-1\right]\frac{\partial^2}{\partial x_i^2} f(\mathsf{Y}_i^\varepsilon(\tau))\,|\,\mathscr{F}_s\right\}d\tau=0,$$
$$0<s<t<\infty. \tag{33}$$

Granted (33), taking a convergent subsequence, still denoted by $\mathsf{Y}_i^\varepsilon(\tau)$, and passing to the limit we have

$$\lim_{\varepsilon\to 0}\mathbf{E}\left\{f(\mathsf{Y}_i^\varepsilon(t))-\tfrac{1}{2}\int_0^t \Delta f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau\,|\,\mathscr{F}_s\right\}$$
$$=\lim_{\varepsilon\to 0}\left\{f(\mathsf{Y}_i^\varepsilon(s))-\tfrac{1}{2}\int_0^s \Delta f(\mathsf{Y}_i^\varepsilon(\tau))\,d\tau\right\}.$$

Namely, the limit martingale $\mathsf{Y}_i^0(t)$ is such that

$$f(\mathsf{Y}_i^0(t))-\tfrac{1}{2}\int_0^t \Delta f(\mathsf{Y}_i^0(\tau))\,d\tau$$

is also a martingale. Thus $\mathsf{Y}_i^0(t)$ is the standard Brownian motion. The proof of Theorem 1 would be complete once (33) is proved.

With $\mathsf{Y}_i^\varepsilon(\tau)$ in (33) replaced by $X_i^\varepsilon(\tau)$ the limit can be calculated as follows. Let

$$g_i^\varepsilon(\mathbf{y})=\varepsilon\sigma_\varepsilon^{-1}(e_i)\,|\nabla\rho_i^\varepsilon(\mathbf{y})|^2-1$$

which is a zero-mean, periodic function with ε-*uniformly* bounded L^1-norm. By the invariance of the Lebesgue measure under the evolution the limit on the left side of (33) becomes

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_s^t \left\langle g_i^\varepsilon(\mathbf{y})\frac{\partial^2}{\partial x_i^2} f(\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}(e_i)}\,\mathbf{y})\right\rangle d\tau$$
$$=\lim_{\varepsilon\to 0}\frac{1}{2}(t-s)\left\langle g_i^\varepsilon(\mathbf{y})\frac{\partial^2}{\partial x_i^2} f(\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}(e_i)}\,\mathbf{y})\right\rangle$$
$$=\lim_{\varepsilon\to 0}\frac{1}{2}(t-s)\langle g_i^\varepsilon\rangle\frac{\partial^2}{\partial x_i^2} f(0)$$
$$=0.$$

Here we have used the fact that $\lambda_\varepsilon^{-1}\sqrt{\sigma_\varepsilon^{-1}}\,\mathbf{y}$ tends to zero for any $\mathbf{y}$ in a fixed period cell in view of (30).

From (31) we see that the limiting Brownian motion has the total variance dt and hence it must be the d-dimensional standard Brownian motion. The proof is complete.

5. BOUNDS ON TIME SCALES

In this section we establish some estimates on the dissipation and the martingale times to clarify the implications of Theorem 1. As before, the vector field $\boldsymbol{u}(\mathbf{x})$ is assumed to be continuously differentiable and divergence free.

First we introduce some asymptotics notations: Given two sequences of positive numbers $a_\varepsilon, b_\varepsilon$, we say that a_ε is *less than or equivalent to* b_ε, denoted by

$$a_\varepsilon \lesssim b_\varepsilon, \qquad \text{if} \quad \limsup_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} < \infty, \tag{34}$$

that a_ε is *larger than or equivalent to* b_ε, denoted by

$$a_\varepsilon \gtrsim b_\varepsilon, \qquad \text{if} \quad \liminf_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} > 0, \tag{35}$$

and that a_ε is *equivalent to* b_ε, denoted by

$$a_\varepsilon \sim b_\varepsilon,$$

if both (34) and (35) hold. Also, a_ε is *much less than* b_ε, denoted by

$$a_\varepsilon \ll b_\varepsilon, \qquad \text{if} \quad \limsup_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} = 0$$

and a_ε is *much larger than* b_ε, denoted by

$$a_\varepsilon \gg b_\varepsilon, \qquad \text{if} \quad \liminf_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} = \infty.$$

The following upper bound on t_{diss} follows from the energy estimate.

PROPOSITION 2. *As ε tends to zero,*

$$t_{diss} \lesssim \frac{1}{\varepsilon}. \tag{36}$$

Proof. Let f_t denote $\mathscr{P}^\varepsilon_t f$. We have that

$$\frac{\partial f_t}{\partial t} = \frac{\varepsilon}{2} \Delta f_t + \boldsymbol{u} \cdot \nabla f_t. \tag{37}$$

Without loss of generality, we may assume $\langle f \rangle = 0$. Hence $\langle f_t \rangle = 0$.

Multiplying (37) by f_t and integrating we obtain

$$\frac{\partial}{\partial t}\langle f_t^2\rangle = -\varepsilon\langle \nabla f_t \cdot \nabla f_t\rangle \leqslant -c\varepsilon\langle f_t^2\rangle$$

using the Poincaré inequality. Thus, by Gronwall's inequality,

$$\langle f_t^2\rangle \leqslant \langle f^2\rangle\, e^{-c\varepsilon t},$$

or, equivalently,

$$\|\mathscr{P}_t^\varepsilon\|_{2\to 2} = e^{-N_\varepsilon(t)} \leqslant e^{-c\varepsilon t}.$$

This proves the proposition in view of the definition of the rate function.

The bound (36) is sharp for nonergodic flows as stated in the following proposition.

PROPOSITION 3. *If the group of diffeomorphisms generated by* $\mathbf{u}(\mathbf{x})$ *is* not *ergodic on the torus and if there exists an invariant measure with* C^2 *variable density* $m(\mathbf{x})$ *then*

$$\|(\mathscr{L}^\varepsilon)^{-1}\|_{2\to 2} \sim \frac{1}{\varepsilon} \tag{38}$$

and

$$t_{diss} \sim \frac{1}{\varepsilon} \tag{39}$$

as ε *tends to zero.*

Proof. First let us make a simple observation

$$C\varepsilon t \leqslant N_\varepsilon(t), \qquad \text{uniformly in } t \in [0, \infty) \tag{40}$$

for some positive constant C independent of ε and t. Indeed, by the Trotter product formula

$$\mathscr{P}_t^\varepsilon = \lim_{n\to\infty} [(e^{\frac{t\varepsilon}{2n}\Delta})(e^{\frac{t}{n}u\cdot\nabla})]^n \tag{41}$$

we have

$$\|\mathscr{P}_t^\varepsilon\|_{2\to 2} \leqslant \lim_{n\to\infty} \|e^{\frac{t\varepsilon}{2n}\Delta}\|_{2\to 2}^n\, \|e^{\frac{t}{n}u\cdot\nabla}\|_{2\to 2}^n \tag{42}$$

$$= \lim_{n\to\infty} \|e^{\frac{t\varepsilon}{2n}\Delta}\|_{2\to 2}^n \tag{43}$$

$$\leqslant e^{-C\varepsilon t}. \tag{44}$$

The constant C can be chosen as the first nonzero eigenvalue of $-\Delta/2$. Here we have used the fact that the divergence free vector field $\boldsymbol{u}$ generates a unitary group $e^{t\boldsymbol{u}\cdot\nabla}$ so that $\|e^{(t/n)\,\boldsymbol{u}\cdot\nabla}\|_{2\to 2}=1$.

We turn to (38). In the definition of the resolvent norm

$$\|(\mathscr{L}^{\varepsilon})^{-1}\|_{2\to 2}=\left(\inf_{f\in C^2}\frac{|\mathscr{L}^{\varepsilon}f|_2}{|f|_2}\right)^{-1} \tag{45}$$

using

$$f(\mathbf{x})=m(\mathbf{x})-\langle m\rangle \tag{46}$$

we get a lower bound

$$\|(\mathscr{L}^{\varepsilon})^{-1}\|_{2\to 2}\geqslant\left(\frac{\left|\frac{\varepsilon}{2}\,\Delta m\right|_2}{|m-\langle m\rangle|_2}\right)^{-1}\geqslant\frac{c}{\varepsilon}. \tag{47}$$

But $\frac{c}{\varepsilon}$ is also an upper bound in view of (40), so the result (38) follows.

To show (39), we analyze as follows. From

$$(\mathscr{L}^{\varepsilon})^{-1}=\int_0^{\infty}dt\,\mathscr{P}_t^{\varepsilon} \tag{48}$$

we have that

$$\begin{aligned}\|(\mathscr{L}^{\varepsilon})^{-1}\|_{2\to 2}&\leqslant\int_0^{\infty}dt\,\|\mathscr{P}_t^{\varepsilon}\|_{2\to 2}\\&\leqslant\int_0^{\infty}dt\,e^{-N_{\varepsilon}(t)}\\&=\int_0^{\tau}dt\,e^{-N_{\varepsilon}(t)}+\int_{\tau}^{\infty}dt\,e^{-N_{\varepsilon}(t)}.\end{aligned} \tag{49}$$

The first integral of the last expression is less than τ, since the integrand is always less than one, and the second integral can be estimated by

$$\begin{aligned}\int_{\tau}^{\infty}dt\,e^{-N_{\varepsilon}(t)}&=\int_0^{\infty}dt\,e^{-N_{\varepsilon}(t+\tau)}\\&\leqslant\int_0^{\infty}dt\,e^{-N_{\varepsilon}(t)}e^{-N_{\varepsilon}(\tau)}\\&=e^{-N_{\varepsilon}(\tau)}\int_0^{\infty}dt\,e^{-N_{\varepsilon}(t)}\\&\leqslant\frac{c}{\varepsilon}\,e^{-N_{\varepsilon}(\tau)}.\end{aligned} \tag{50}$$

Here we have used the superadditivity of the rate function and the bound (40). Thus, we have

$$\tau \geqslant \|(\mathscr{L}^{\varepsilon})^{-1}\|_{2\to 2} - \frac{c}{\varepsilon}\, e^{-N_\varepsilon(\tau)} \tag{51}$$

for any τ. Any $\tau \gg t_{diss}$ in (51) would make the second term $o(\varepsilon^{-1})$, while the first term is always of the order ε^{-1}; i.e., for any $\tau \gg t_{diss}$ we must also have $\tau \gtrsim \varepsilon^{-1}$. Thus $t_{diss} \gtrsim \varepsilon^{-1}$ and therefore, in view of the upper bound (36), $t_{diss} \sim \varepsilon^{-1}$. This completes the proof.

However t_{diss} can be substantially smaller than ε^{-1} as is the case for a discrete-time system considered in Section 6.

A similar bound on t_{mart} holds, namely

PROPOSITION 4. *As ε tends to zero,*

$$t_{mart} \lesssim \frac{1}{\varepsilon}. \tag{52}$$

This follows easily from the Poincaré inequality

$$\sigma_\varepsilon(e_i) = \varepsilon + \varepsilon\langle \nabla\chi_i^\varepsilon \cdot \nabla\chi_i^\varepsilon\rangle \geqslant c\varepsilon\langle(\chi_i^\varepsilon)^2\rangle. \tag{53}$$

The bound (52) is optimal in general (see the example of the open-channel flow in Section 6).

Now we explicate the condition (10) of Theorem 1 and give an alternative, more precise upper bound on the diffusion time t_{diff}.

By superadditivity of exponent $N_\varepsilon(t)$, we have that

$$N_\varepsilon\left(ct_{diss}\log\frac{1}{\varepsilon}\right) \geqslant cN_\varepsilon(t_{diss})\log\frac{1}{\varepsilon} \geqslant \frac{c}{2}\log\frac{1}{\varepsilon}.$$

So, if $\lambda_\varepsilon^2 \gg t_{diss}\log 1/\varepsilon$, then $N_\varepsilon(\lambda_\varepsilon^2) \gg \log\frac{1}{\varepsilon}$ and (9) holds true. Thus, it follows from Theorem 1 that

$$t_{diff} \lesssim t_{mart} \vee \left(t_{diss}\log\frac{1}{\varepsilon}\right).$$

That is, condition (10) introduces no more than a logarithmic factor $-\log\varepsilon$ to (9).

On the other hand, by Propositions 2 and 4, any $\lambda_\varepsilon^2 \gg 1/\varepsilon$ would satisfy (9) and (10) of Theorem 1 and hence

$$t_{diff} \lesssim \frac{1}{\varepsilon}.$$

Thus, we have shown

PROPOSITION 5.

$$t_{diff} \lesssim \frac{1}{\varepsilon} \wedge \left(t_{mart} \vee t_{diss} \log \frac{1}{\varepsilon} \right).$$

When $t_{diss} \ll 1/\varepsilon$, due to the likely overestimate in (53) (if so, then $t_{mart} \ll \varepsilon^{-1}$), the diffusion time may be much smaller than ε^{-1}. Some examples for which $t_{mart} \ll 1/\varepsilon$ or $t_{diss} \ll 1/\varepsilon$ are given in the next section.

In general, $t_{diss}, t_{mart} \lesssim t_{diff}$ but there is no definite ordering between t_{diss} and t_{mart} as we will see in the examples of the next section.

6. EXAMPLES

We may summarize our results in the following scenario: On $O(1)$ time scale, the noise induces a small perturbation to the dynamical system; on the martingale time, the noisy system begins to behave like a martingale, with highly unpredictable increments; in the meantime, it may or may not have passed the dissipation time and when it does, the increments of the martingale are approximately stationary and the process approaches the asymptotic state of pure diffusion.

In this section, we examine some examples in two dimensions to illustrate how t_{diss}, t_{mart} are affected by the flow structures. We refer the readers to [7] for the more complete discussion of the effective diffusivity σ_ε.

In two dimensions, when the velocity $\boldsymbol{u}(\mathbf{x})$ has zero mean, there exists a periodic Hamiltonian $H(\mathbf{x})$ such that

$$\nabla^\perp H(\mathbf{x}) = \boldsymbol{u}(\mathbf{x})$$

and the contours of $H(\mathbf{x})$ are exactly the trajectories.

There are two typical flow structures for in two dimensions: one is represented by the cellular flow (Fig. 1) whose stream function is

$$H(\mathbf{x}) = \sin(2\pi x) \cdot \sin(2\pi y), \qquad \mathbf{x} = (x, y). \tag{54}$$

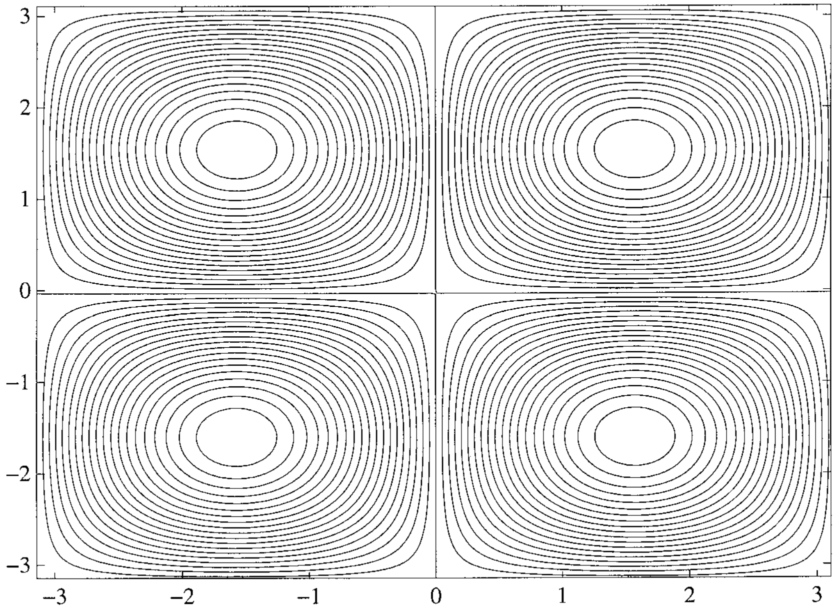

FIG. 1. Cellular Flow

The other is represented by the open channel flow (Fig. 2) given by

$$H_\delta(\mathbf{x}) = \sin(2\pi x)\cdot\sin(2\pi y) + \delta\cos(2\pi x)\cdot\cos(2\pi y). \tag{55}$$

For the cellular flow, it is known that

$$\sigma_\varepsilon(e) \asymp c^*\sqrt{\varepsilon}, \qquad \forall e,$$

as $\varepsilon \to 0$, where the constant c^* can be exactly determined. Because all streamlines in the cellular flow are homologous to a point on the torus (they are close streamlines in the plane) it can be shown by the maximum principle for $\mathscr{L}^\varepsilon$ that the corrector is bounded uniformly in ε. So the martingale and dissipation times are

$$t_{mart} \sim \frac{1}{\sqrt{\varepsilon}}, \qquad t_{diss} \sim \frac{1}{\varepsilon}.$$

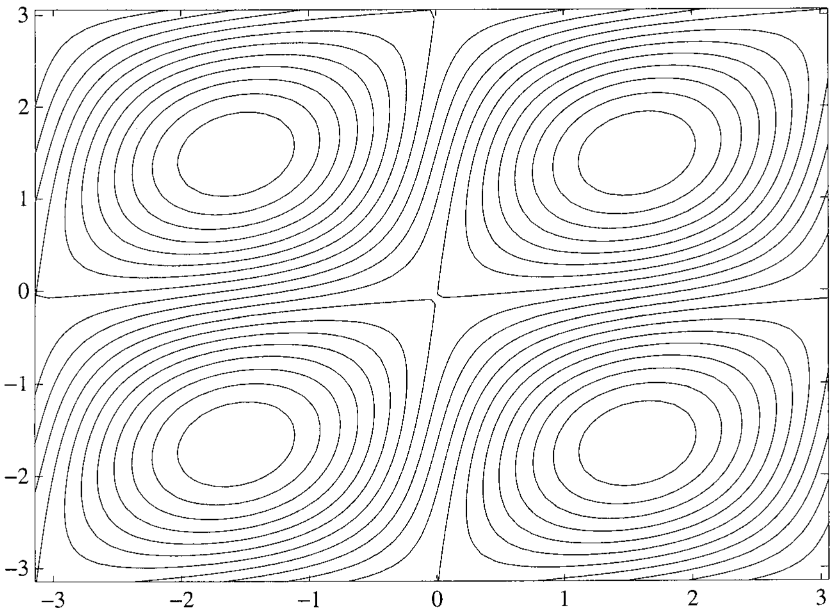

FIG. 2. Open Channel Flow

Thus, by Proposition 5, the diffusion time t_{diff} is

$$t_{diff} \sim \frac{1}{\varepsilon}.$$

For the cellular flow, the martingale time is much smaller than the dissipation time and, thus, opens the possibility for transient anomalous diffusion (see [17] for a physical explanation). This is not the case for the open channel flow or baker's map as we shall see below.

For the open channel flow (55), it is known that

$$\sigma_\varepsilon(e_1) \asymp \frac{\delta^3}{3\varepsilon}, \qquad \sigma_\varepsilon(e_2) \asymp \frac{\varepsilon}{\delta}$$

where $e_1 = 1/\sqrt{2}\,(1, 1)$, $e_2 = 1/\sqrt{2}\,(-1, 1)$ are parallel and orthogonal, respectively, to that of the open channels.

The perturbation in (55) opens up the separatrices and creates streamlines that are not homologous to a point on the torus. As a result the corrector $\chi^\varepsilon(x, y) = (\chi_1^\varepsilon(x, y), \chi_2^\varepsilon(x, y))$ becomes singular as $\varepsilon \to 0$ in the $\boldsymbol{e}_1$ direction

$$\langle |\chi_1^\varepsilon|^2 \rangle \sim \frac{1}{\varepsilon^2}, \qquad \langle |\chi_2^\varepsilon|^2 \rangle \sim 1.$$

Hence,

$$t_{mart} \sim \frac{1}{\varepsilon}, \qquad t_{diff} \sim \frac{1}{\varepsilon}.$$

Let us now consider the discrete-time system defined by baker's map on the period cell $[0, 1]^2$:

$$F(x, y) = \begin{cases} (2x, \frac{1}{2} y) & \text{if } \ 0 \leqslant x < 1/2 \\ (2x-1, \frac{1}{2}(y+1)) & \text{if } \ 1/2 \leqslant x < 1. \end{cases}$$

Define the unperturbed transition operator $\mathscr{P}_n$, $n = 0, 1, 2, \ldots$ as

$$\mathscr{P}_n \equiv (\mathscr{P}_1)^n, \qquad \text{with} \quad \mathscr{P}_1 f(\mathbf{x}) = f(F(\mathbf{x})). \tag{56}$$

We add noise to (56) by convoluting it with a heat kernel G_ε of variance ε

$$\mathscr{P}_t^\varepsilon = (\mathscr{P}_1^\varepsilon)^t, \qquad \text{with} \quad \mathscr{P}_1^\varepsilon f(\mathbf{x}) = G_\varepsilon * (f \circ F)(\mathbf{x}).$$

It can be shown that the dissipation time t_{diss} defined as

$$t_{diss} = \min\{n \mid \|\mathscr{P}_n^\varepsilon\|_{2\to 2} \leqslant 1/2\}$$

is shorter than any positive power of ε^{-1} due to the fast mixing property of baker's map. Also, it has been shown that $t_{diss} \sim \ln \frac{1}{\varepsilon}$ for any ergodic toral automorphisms [5].

The correctors χ_i^ε, $i = 1, 2, \ldots, d$, satisfy the equation

$$(P_1^\varepsilon - I)\, \chi_i^\varepsilon(\mathbf{x}) = x_i - G_\varepsilon * F(\mathbf{x})$$

or, equivalently, the harmonic coordinates $\rho_i^\varepsilon(\mathbf{x}) \equiv \chi_i^\varepsilon(\mathbf{x}) + x_i$ satisfy

$$\mathscr{P}_1^\varepsilon \rho_i^\varepsilon(\mathbf{x}) = \rho_i^\varepsilon(\mathbf{x}), \qquad i = 1, 2, \ldots, d.$$

We define the effective diffusivity σ_ε by

$$\sigma_\varepsilon(\boldsymbol{e}_i) = \langle (\mathscr{P}_1^\varepsilon - I)(\rho_i^\varepsilon)^2 \rangle \tag{57}$$

which is a natural extension of (18) to discrete-time systems in view of the following identity in the time-continuous case

$$\mathscr{L}^{\varepsilon}(\rho_i^{\varepsilon})^2 = 2\rho_i^{\varepsilon}\mathscr{L}^{\varepsilon}\rho_i^{\varepsilon} + \varepsilon\,\nabla\rho_i^{\varepsilon}\cdot\nabla\rho_i^{\varepsilon} = \varepsilon\,\nabla\rho_i^{\varepsilon}\cdot\nabla\rho_i^{\varepsilon}.$$

The martingale time t_{mart} is then defined as

$$t_{mart} = \max\{n \mid n \leqslant \sigma_{\varepsilon}^{-1}(e_i)\langle(\chi_i^{\varepsilon})^2\rangle,\ i = 1, 2, ..., d\}.$$

It is known that the correctors χ_i^{ε} are uniformly bounded in L^2 norm

$$\langle|\chi_i^{\varepsilon}|^2\rangle \sim 1, \qquad \forall i$$

and the effective diffusivity behaves similar to that of the cellular flow

$$\sigma_{\varepsilon} \sim \sqrt{\varepsilon}$$

(see [4]). Hence the martingale time

$$t_{mart} \sim \frac{1}{\sqrt{\varepsilon}}$$

which is much larger than the dissipation time. Thus, analogous to Proposition 5, we have for noisy baker's system that

$$t_{diff} \lesssim \frac{1}{\sqrt{\varepsilon}}.$$

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